**Prop** (Uniqueness of limits). In Hausdorff topological spaces, limits of sequences are unique (if exist).

**Proof.** Let $x \neq x'$ in $X$ and suppose $x_n \to x$. We show that $x_n \not\to x'$. By Hausdorffness, there are disjoint open $U \ni x$ and $V \ni x'$. Because $x_n \to x$, we have that $\forall^\infty_n \ x_n \in U$. Hence $\forall^\infty_n \ x_n \notin V$, so $x_n \not\to x'$. $\square$
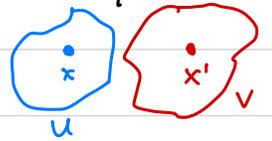
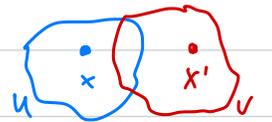**Prop.** Let $X$ be a $1^{st}$ ctbl top. space. If limits of sequences are unique then $X$ is Hausdorff.

**Proof.** Let $x \neq x'$ be in $X$. Suppose $x$ and $x'$ don't admit disjoint open neighbourhoods, i.e. for each $U \ni x$ and $V \ni y$ open, $U \cap V \neq \emptyset$. Fix ctbl neighbourhood bases $\{U_n\}_{n \in \mathbb{N}}$ and $\{V_n\}_{n \in \mathbb{N}}$ for $x$ and $x'$, respectively. By replacing $U_n$ with $\bigcap_{i \le n} U_i$, and $V_n$ with $\bigcap_{i \le n} V_i$, we may assume WLOG that $\{U_n\}_{n \in \mathbb{N}}$ and $\{V_n\}_{n \in \mathbb{N}}$ are decreasing. For each $n \in \mathbb{N}$, choose (AC) $x_n \in U_n \cap V_n$ to obtain a sequence $(x_n)_{n \in \mathbb{N}}$. It is now easy to see that $x_n \to x$ and $x_n \to x'$. Indeed, it is enough to check convergence for $U_n$ and $V_n$, i.e. check that for each fixed $n \in \mathbb{N}$, $\forall^\infty_m \ x_m \in U_n$ and $x_m \in V_n$, but this is true for all $m \ge n$ by construction and the fact that the $U_n$ and $V_n$ are decreasing. $\square$

We now recall terms like neighbourhood, closure, and boundary.

**Def.** Let $X$ be top. space. A **neighbourhood** of a point $x \in X$ is a set $P \ni x$ s.t. $\exists$ open set $U$ with $x \in U \subseteq P$.
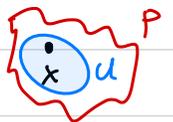
**Def.** Let $X$ be a top. space. The **interior** of a set $P \subseteq X$ is the largest open set $U \subseteq P$, namely the union of all open $V \subseteq P$. This is denoted by $\mathrm{int}(P)$ or $\mathrm{int}\, P$.

**Def.** Let $X$ be a top. space. The **closure** of a set $P \subseteq X$, denoted $\bar{P}$, is the smallest closed

set $C \supseteq P$, namely, the intersection of all closed sets containing $P$. We say that a point $x \in X$ **adheres** to $P$ if every open neighbourhood of $x$ intersects $P$.

**Prop.** For any top. space $X$ and $P \subseteq X$, $\bar{P} = \{x \in X : x \text{ adheres to } P\}$.

**Proof.** Literally the same as for metric spaces (the proof only used open sets). □

**De Morgan for interior/closure.** Let $X$ be a top. space and $P \subseteq X$.

(a) $\text{int}(P) = \left(\overline{(P^c)}\right)^c$.

(b) $\bar{P} = \text{int}(P^c)^c$.

**Examples.** In $\mathbb{R}$.

(a) $\text{int}([0,1)) = (0,1)$ and $\text{int}([0,1] \cup \{2,3\}) = (0,1)$.

(b) For the standard Cantor set $C \subseteq [0,1]$, $\text{int}(C) = \emptyset$.

(c) $\text{int}(\mathbb{Q}) = \emptyset$ because every nonempty open interval is uncountable while $\mathbb{Q}$ is ctbl.

(d) $\text{int}(\mathbb{R} \setminus \mathbb{Q}) = \emptyset$ because $\mathbb{Q}$ is dense.

**Def.** Let $X$ be a top. space. The **boundary** of a set $P \subseteq X$, denoted $\partial P$, is the set $\bar{P} \cap \overline{(P^c)}$. Equivalently, $\partial P = \{x \in X : x \text{ adheres to both } P \text{ and } P^c\}$.

**Examples.** In $X = \mathbb{R}$.

(a) $\partial([0,1] \cup \{2,3\}) = \{0,1,2,3\}$

(b) Let $C \subseteq [0,1]$ be the standard Cantor set. $\partial C = C$ because $\text{int}(C) = \emptyset$, so $\mathbb{R} \setminus C$ is dense, hence $\overline{(\mathbb{R} \setminus C)} = \mathbb{R}$ so $\mathbb{R} \cap C = C$.

(c) $\partial \mathbb{Q} = \mathbb{R}$ because both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$.

(d) $\partial(\mathbb{R} \setminus \mathbb{Q}) = \mathbb{R}$ for the same reason.

**Obs.** For a top. space $X$ and $P \subseteq X$, $\partial P = X \setminus (\text{int}(P) \cup \text{int}(P^c))$.

Recall that in metric spaces, a point $x$ is in the closure of a set $P$ if and only if there is a sequence $(p_n) \subseteq P$ converging to $x$. The implication $\Leftarrow$ holds in all top spaces while $\Rightarrow$ requires 1ˢᵗ ctbl.

**Prop.** Let $X$ be a top. space and $P \subseteq X$. Let $x \in X$.

(a) If $\exists$ sequence $(p_n) \subseteq P$ converging to $x$, then $x \in \overline{P}$.

(b) If $X$ is 1ˢᵗ ctbl and $x \in \overline{P}$, then $\exists$ sequence $(p_n) \subseteq P$ converging to $x$.

**Proof.** (a) If $p_n \to x$ then for each open $U \ni x$, $U \cap P$ contains the tail of $(p_n)$, hence is nonempty. Thus, $x$ adheres to $P$, so $x \in \overline{P}$.

(b) Same as for metric spaces replacing $B_{1/n}(x)$ with an arbitrary decreasing ctbl neighbourhood basis for $x$. □

**Remark.** It seems like sequences capture the topology only in 1ˢᵗ spaces, while in general top. spaces the right notion is that of a **net**. We'll discuss this notion later if time permits.
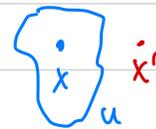
## Separation axioms.

Let $X$ be a top space. We say that $X$ is

○ $T_0$ if for any $x \neq x'$ in $X$, either $\exists$ open $U \ni x$ s.t. $x' \notin U$ or $\exists$ open $V \ni x'$ s.t. $x \notin V$.
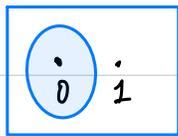


or
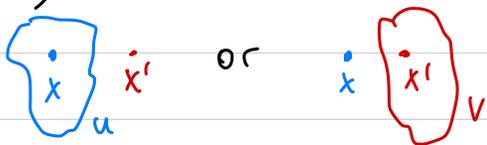
○ $T_1$ (= singletons are closed) if for any $x \neq x'$ $\exists$ open $U \ni x$ s.t. $x' \notin U$.
Equivalently, $\exists$ open $U \ni x$ and open $V \ni x'$ s.t. $x' \notin U$ and $x \notin V$.
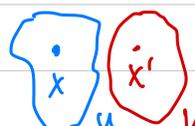


**Prop.** A top. space $X$ is $T_1$ $\Longleftrightarrow$ every singleton $\{x\}$ is a closed set.
**Proof.** HW

The half open topology, i.e. [figure: $\cdot$ 0 enclosed, $\cdot$ 1], so $X = \{0,1\}$ and the open sets are $\emptyset, X, \{0\}$.

Indeed, $\nexists$ open $V \ni 1$ that does not contain $0$, in other words, $\{0\}$ is not a closed set. The constant $0$ sequence $x_n := 0$ converges to both $0$ and $1$.

○ $T_2$ ($=$ Hausdorff) if for any $x \neq x'$ in $X$ $\exists$ disjoint open $U \ni x$ and $V \ni x'$. [figure: $\cdot x$ in $U$, $\cdot x'$ in $V$]

Example of $T_1$ but not $T_2$. Let $\tau$ be the cofinite top. on an infinite set $X$, i.e. a set $U \subseteq X$ is open $\iff$ $X \backslash U$ is finite. This is $T_1$ because for any $x \neq x'$ in $X$, the set $U := X \backslash \{x'\}$ is an open neighbourhood of $x$ not containing $x'$. However, this top is not Hausdorff because the intersection of any two non-empty open sets $U, V$ is still a cofinite set, hence nonempty.

○ $T_3$ ($=$ regular) if it is $T_1$ and for each $x \in X$ and closed set $C \subseteq X$ with $x \notin C$ $\exists$ disjoint open $U \ni x$ and $V \supseteq C$.
[figure: $\cdot x$ in $U$, $C$ in $V$]

Note. Because singletons are closed in $T_1$, $T_3 \Rightarrow T_2$.